

Modeling Heterogeneous Network Interference with Using Poisson Point Processes

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Abstract—Cellular systems are becoming more heterogeneous with the introduction of low power nodes including femtocells, relays, and distributed antennas. Unfortunately, the resulting interference environment is also becoming more complicated, making evaluation of different communication strategies challenging in both analysis and simulation. This paper suggests a simplified interference model for heterogeneous networks. Leveraging recent applications of stochastic geometry to analyze cellular systems, this paper proposes to analyze downlink performance in a fixed-size cell in what is called the hybrid approach. The interference field consists of a contribution from out-of-cell interferers modeled as a superposition of marked Poisson point processes outside a guard region and a cross-tier contribution modeled as a marked Poisson point process. The guard region is used to avoid association issues that would occur if an interferer were located at the cell edge. Bounding the interference power as a function of distance from the cell center, the total interference is characterized through its Laplace transform. An equivalent marked process is proposed for the out-of-cell interference under additional assumptions. To facilitate simplified calculations, the interference distribution is approximated using the Gamma distribution with second order moment matching. The Gamma approximation simplifies calculation of the success probability and ergodic rate, incorporates small-scale and large-scale fading, and works with co-tier and cross-tier interference. Simulations show that the guard region can be tuned to give a reasonable match with performance in a hexagonal grid, and that the Gamma approximation provides a good fit over most of the cell.

I. INTRODUCTION

Cellular network deployment is taking on a massively heterogeneous character as a variety of infrastructure is being deployed including macro, pico, and femto base stations [1], as well as fixed relay stations [2] and distributed antennas [3]. A major challenge in deploying heterogeneous cellular networks is managing interference. The problem is compounded by the observation that the deployment of much of the small cell infrastructure will be demand based and more irregular than traditional infrastructure deployments [4]. Further, as urban areas are built out, even macro and micro base station infrastructure is becoming less like points on a hexagonal lattice and more random [5]. Consequently, the aggregate interference environment is more complicated to model and evaluating the performance of communication techniques in the presence of heterogeneous interference is challenging.

Stochastic geometry can be used to develop a mathematical framework for dealing with interference [6]–[9]. With stochastic geometry, interferer locations are distributed according to a point process, often the homogeneous Poisson Point Process

(PPP) for its tractability. PPPs have been used to model co-channel interference from macro cellular base stations [5], [10]–[13], cross-tier interference from femtocells [4], [14], [15], co-channel interference in ad hoc networks [7]–[9], [16], and as a generic source of interference [17]–[19].

Modeling co-channel interference from other base stations as performed in [5], [10], [12], [13], [20] is a good starting point for developing insights into heterogeneous network interference. In [10] PPPs are used to model various components of a telecommunications network including subscriber locations, base station locations, as well as network infrastructure leveraging results on Voronoi tessellation. In [12], [20] a cellular system with PPP distribution of base stations, called a shotgun cellular system, is shown to lower bound a hexagonal cellular system in terms of certain performance metrics and to be a good approximation in the presence of shadow fading. In [5], a comprehensive analysis of a cellular system with a PPP is presented, where key system performance metrics like coverage probability and average rate are computed by averaging over all deployment scenarios. The approach in [5] is quite powerful, leading to closed form solutions for some special choices of parameters. In [13] the outage and the handover probabilities in a PPP cellular network are derived for both random general slow fading and Rayleigh fading, where mobiles are attached to the base station that provides the best mean signal power. Recent work has considered extensions of [5], [12] to multi-tier networks. For example [21] extends [12] to provide some results on the signal-to-interference ratio in multi-tier networks while [22] extends [5] to provide remarkable simple expressions for the success probability in multi-tier networks. From our perspective, the main drawbacks of the approach in [5], [10], [12], [13], [20] for application to existing systems is that performance is characterized for an entire system not for a given cell. As cellular networks are already built out, a cellular provider will often want to know the performance they achieve in some given cells by adding additional infrastructure (and thus interference) in the rest of the network. It is also challenging to incorporate more complex kinds of heterogeneous infrastructure like fixed relays or distributed antennas into the signal and interference models.

Models for co-channel interference from Poisson Point Processes in general wireless settings has been considered in prior work [16]–[19], [23]. References [17]–[19] derive models for the complex noise distribution. These models assume that the noise changes on a sample-by-sample basis and are suitable for deriving optimum signal processing algorithms for example optimum detectors. Other work, see e.g. [16], [23] and the references therein, provide models for the noise power distribution. Most system-level analysis work [5], [10], [12], [13],

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[20] focuses on the noise power distribution. They assume that the interference distribution conditioned on the noise power is Gaussian and thus characterize performance based on quantities like the signal-to-interference ratio or the signal-to-interference-plus-noise ratio, which are not appropriate for other noise distributions [17]–[19]. This is reasonable because cellular systems tend to have structured transmissions in time and frequency, the interfering sets are likely to be constant over a coding block. Consequently we focus on characterizing and employing the noise power distribution in our work.

In this paper we propose a simplified interference model for heterogeneous networks called the hybrid approach. The key idea is to evaluate performance in a fixed-size cell considering co-channel interference outside of a guard region and cross-tier interference from within the cell. Different sources of interference are modeled as marked Poisson point processes where the mark distribution includes a contribution from small-scale fading and large-scale fading. The guard region is used to avoid association issues that would occur if co-channel interferer were located at the cell edge. Based on simulations, an optimum guard region size of two-thirds the cell radius is suggested to achieve good match with hexagonal interferer locations. For co-channel interference, an upper bound on the interference power distribution is characterized through its Laplace transform as a function of distance from the cell center for the case of downlink transmission, after lower bounding the size of the exclusion region. When there are multiple types of co-channel interference, the distribution power can be further upper bounded by a single marked process that is a function of the mark parameters, densities, and minimum guard radius from each interference process. One of the key advantages of the proposed model is that no explicit user association scheme needs to be specified. Most prior work on heterogeneous cellular networks analysis using stochastic geometry derives the SINR distribution assuming a specific user assignment scheme. In [24], the downlink SINR distribution from the serving BS to an arbitrarily-located user is derived when the candidate serving BS at each tier is the nearest to the user. In [25] the user is attached to the BS providing the strongest average received signal, whereas in [22], [26], user association based on maximum SINR connectivity is considered. In the hybrid model, users can be arbitrarily distributed in the cell of interest without requiring any assumption on the user association. This allows us to analyze the cell performance and calculate per-user performance metrics under various user distributions.

To facilitate simplified calculations, we use approximations based on the Gamma distribution. We model the small-scale fading component using the Gamma distribution as this incorporates single user and multiple user signaling strategies with Rayleigh fading as special cases. We approximate the product of the Gamma distributed small-scale fading component and the log-normal large-scale fading component as a Gamma distributed random variable, based on prior work [27]. We then approximate the interference distribution using the Gamma distribution with second order moment matching; the moments are finite because we use a nonsingular path-loss model. Using the Gamma approximation, we present simplified expressions

for the success probability and the average rate, assuming that the system is interference limited, i.e. our calculations are based on the signal-to-interference ratio. Simulations show that the Gamma approximation provides a good fit over most of the cell, with larger error towards the cell edge.

Our mathematical approach follows the approach of [5] and leverages basic results on PPPs [8], [9], [23]. Compared with [5], we consider a fixed-size cell and do not evaluate performance in a Voronoi tessellation of randomly distributed interferers. We also do not calculate system-wide performance measures, rather we focus on performance for a fixed cell of interest. We demonstrate how to employ the proposed fixed cell analysis in a heterogeneous network consisting of mixtures of the different kinds of infrastructure. The advantage of our approach is that more complex types of communication and network topologies can be analyzed in a given target cell and key insights on the performance of advanced transmission techniques with heterogeneous interference using stochastic geometry can be obtained, at the expense of generality. Compared with our preliminary results reported in [28], in this paper we deal with more general fading distributions and use the Gamma approximation to simplify the calculation of the success probability and the ergodic rate. Guard zones have been used in other work on interference models e.g. [19], [29]. In [29] they are used to evaluate the transmission capacity in contention-based ad hoc networks not cellular networks while in [19] they are considered in the derivation of the baseband noise distribution but not the noise power.

II. MATHEMATICAL PRELIMINARIES

In this paper we make frequent use of Gamma random variables. We use them to model the small-scale fading distribution, to approximate the product of the small-scale and log-normal fading distribution, and to approximate the interference power. In this section we provide some background on Gamma random variables to make the paper more accessible. Proofs are omitted where results easily follow or are well known in the literature.

Definition 1: A Gamma random variable with finite shape $k > 0$ and finite scale $\theta > 0$, denoted as $\Gamma[k, \theta]$ has probability distribution function

$$f_X(x) = x^{k-1} \frac{e^{-x/\theta}}{\theta^k \Gamma(k)} \quad (1)$$

for $x \geq 0$. The cumulative distribution function is

$$F_X(x) = \frac{\gamma(k, x/\theta)}{\Gamma(k)} \quad (2)$$

where $\gamma(a, b) = \int_0^b t^{a-1} e^{-t} dt$ is the lower incomplete gamma function. The first two moments and the variance are

$$\mathbb{E}X = k\theta \quad \mathbb{E}X^2 = k(1+k)\theta^2 \quad \text{var } X = k\theta^2. \quad (3)$$

The scale terminology comes from the fact that if X is $\Gamma[k, \theta]$ then with scalar $\alpha > 0$, αX is $\Gamma[k, \alpha\theta]$.

The Gamma distribution is relevant for wireless communication because it includes several channel models as a special case. For example if Z is a circularly symmetric complex Gaussian with $\mathcal{N}(0, 1)$ (variance $1/2$ per dimension) then

$X = |Z|^2$ is $\Gamma[1, 1]$, which corresponds to Rayleigh channel assumption. If X has a Chi-square distribution with $2N$ degrees of freedom, used in the analysis of diversity systems, then X is also $\Gamma[N, 2]$. If X is Nakagami distributed [30] with parameters μ and ω then $Y = X^2$ has a $\Gamma[k, \theta]$ distribution with $k = \mu$ and $\theta = \omega/\mu$. Consequently the Gamma distribution can be used to represent common fading distributions.

Of relevance for computing rates, the expected value of the log of a Gamma random variable has a simple form as summarized in the following Lemma.

Lemma 2 (Expected Log of a Gamma Random Variable): If X is $\Gamma[k, \theta]$ then $\mathbb{E} \ln X = \psi(k) + \ln(\theta)$ where $\psi(k)$ is the digamma function.

The digamma function is implemented in many numerical packages or for large x its asymptotic approximation can be used $\psi(x) = \ln(x) + 1/x + \mathcal{O}(1/x^2)$.

The Gamma distribution is also known as the Pearson type III distribution, and is one of a family of distributions used to model the empirical distribution functions of certain data [31]. Based on the first four moments, the data can be associated with a preferred distribution. As opposed to optimizing over the choice of the distribution, we use the Gamma distribution exclusively because it facilitates calculations and analysis. We will approximate a given distribution with a Gamma distribution by matching the first and second order moments in what is commonly known as moment matching [32]. The two parameters of the Gamma distribution can be found by setting the appropriate moments equal and simplifying. We summarize the result in the following Lemma.

Lemma 3 (Gamma 2nd Order Moment Match): Consider a distribution with $\mu = \mathbb{E}X$, $\mu^{(2)} = \mathbb{E}X^2$, and variance $\sigma^2 = \mu^{(2)} - \mu^2$. Then the distribution $\Gamma[k, \theta]$ with same first and second order moments has parameters

$$k = \frac{\mu^2}{\sigma^2} \quad \theta = \frac{\sigma^2}{\mu}. \quad (4)$$

Note that the shape parameter is scale invariant. For example, the Gamma approximation of αX would have shape $k = \mu^2/\sigma^2$ and scale $\theta = \alpha\sigma^2/\mu$.

We will need to deal with sums of independent Gamma random variables with different parameters. There are various closed-form solutions for the resulting distribution [33], [34] (see also the references in the review article [35]). In this paper we choose the form in [33] as it gives a distribution that is a sum of scaled gamma distributions.

Theorem 4 (Exact Sum of Gamma Random Variables): Suppose that $\{Y_i\}_{i=1}^n$ are independent $\Gamma[k_i, \theta_i]$ distributed random variables. Then the probability distribution function of $Y = \sum_i Y_i$ can be expressed as

$$f_Y(y) = C \sum_{i=0}^{\infty} \frac{c_i}{\Gamma(\rho+i) \theta_{min}^{\rho+i}} y^{\rho+i-1} e^{-y/\theta_{min}} \quad (5)$$

where $\theta_{min} := \min_i \theta_i$, $\rho = \sum_{i=1}^n k_i$, $C = \prod_{i=1}^n (\theta_{min}/\theta_i)^{k_i}$, and c_i is found by recursion from

$$c_{i+1} = \frac{1}{i+1} \sum_{m=1}^{i+1} m \gamma_m c_{i+1-m} \quad (6)$$

and $\gamma_m = \sum_{i=1}^n k_i m^{-1} (1 - \theta_{min}/\theta_i)^m$.

Proof: See [33]. ■

For practical implementation, the infinite sum is truncated; see [33] for bounds on the error. Mathematica code for computing the truncated distribution is found in [36]. In some cases, many terms need to be kept in the approximation to achieve an accurate result. Consequently we also pursue a moment-matched approximation of the sum distribution.

Lemma 5 (Sum of Gammas 2nd Order Moment Match): Suppose that $\{Y_i\}$ are independent Gamma distributed random variables with parameters k_i and θ_i . The Gamma distribution $\Gamma[k_y, \theta_y]$ with the same first and second order moments has

$$k_y = \frac{(\sum_i k_i \theta_i)^2}{\sum_i k_i \theta_i^2} \quad \text{and} \quad \theta_y = \frac{\sum_i k_i \theta_i^2}{\sum_i k_i \theta_i}. \quad (7)$$

Bounds on the maximum error obtained through a moment approximation are known [32] and may be used to estimate the maximum error in the cumulative distribution functions of the Gamma approximation. In related work [37] we show that the approximation outperforms the truncated expression in (17) unless a large number of terms are kept, and therefore conclude that the approximation is reasonable. The approximation is exact in some cases, e.g. if all the shape values are identical (in this case $\theta_i = \theta$ then $k_y = \sum_i k_i = \rho$, $\theta_y = \theta = \theta_{min}$ and $c_i = 0$).

In wireless systems, the zero-mean log-normal distribution is used to model large-scale fluctuations in the received signal power. The distribution is $f_X(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x)^2}{2\sigma^2}}$ where σ is the shadow standard deviation, often given in dB where $\sigma_{dB} = 8.686\sigma$. Typical values of σ_{dB} are between 3dB and 10dB for example in 3GPP [38], [39].

Composite fading channels are composed of a contribution from both small-scale fading and large-scale fading. Consider the random variable for the product distribution HL where H is $\Gamma[k_h, \theta_h]$ and L is log-normal with parameter σ . Related work has shown that the product distribution can be reasonably modeled using a Gamma distribution [27] by matching two of the moments. In our case we find that the first and second central moments gives reasonable performance.

Lemma 6 (Gamma Approximation of Product Distribution): Consider the random variable PHL where P is a constant, H is $\Gamma[k_h, \theta_h]$ and L is log-normal with variance σ . The Gamma random variable with $\Gamma[k_p, \theta_p]$ with the same first-order and second-order moments has

$$k_p = \frac{1}{(1/k_h + 1)e^{\sigma^2} - 1} \quad (8)$$

$$\theta_p = (1 + k_h)\theta_h e^{3\sigma^2/2} - k_h\theta_h e^{\sigma^2/2} \quad (9)$$

Proof: Note that

$$\mathbb{E}HL = \mathbb{E}H \mathbb{E}L = k_h\theta_h e^{\sigma^2/2} \quad (10)$$

$$\text{var}HL = (1 + k_h)k_h\theta_h^2 e^{2\sigma^2} - k_h^2\theta_h^2 e^{\sigma^2}. \quad (11)$$

Substituting into Lemma 3 and simplifying yields (8) and (9). ■

III. DOWNLINK NETWORK MODEL

In the classic model for cellular systems in Fig. 1(a), base stations are located at the centers of hexagons in a hexagonal tessellation and interference is computed in a fixed cell from multiple tiers of interferers. The hexagonal model requires simulation of multiple tiers of interferers and makes analysis difficult. In the stochastic geometry model for cellular systems illustrated in Fig. 1(b), base stations positions follow a PPP and cells are derived from the Voronoi tessellation. Cell sizes are random and performance metrics are computed in an aggregate sense accounting for all base station distributions. Our proposed model, which we call the *hybrid approach*, is illustrated in Fig. 1(c). We consider a cell of fixed shape and size, nominally a circle with radius R_c . The base station locations outside of the fixed cell are modeled according to a PPP and the interference becomes a shot-noise process [40]. Since the edge of the cell is fixed, and is not determined by interference levels, a guard region of radius R_g from the cell edge is imposed around the fixed cell in which no other transmitters can occupy. Performance is evaluated inside the fixed cell accounting for interference sources outside the guard region. In general, R_g is a parameter of the model; our simulations show that $R_g = 2R_c/3$ provides good performance relative to a hexagonal tessellation.

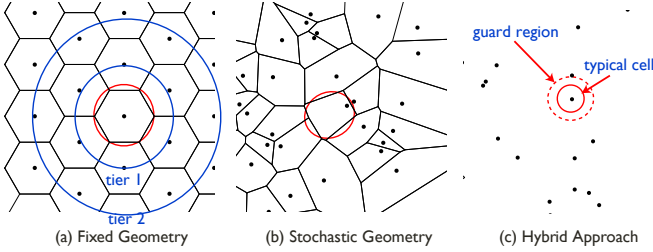


Fig. 1. Models for cellular communication. (a) The common fixed geometry model with hexagonal cells and multiple tiers of interference. (b) A stochastic geometric model where all base stations are distributed according to some 2D random process. (c) The proposed hybrid approach where there is a *fixed cell* of a fixed size surround by base stations distributed according to some 2D random process, possibly with an exclusion region around the cell.

To make the calculations concrete, in this paper we focus on the downlink. We consider the received signal power at distance r from the transmitter presumably located at the cell center. Extension to distributed antennas for example requires a more complex signal model. Let the received signal power be

$$S(r) = P(r)LH, \quad (12)$$

where $P(r) = P_s/\ell(r)$ is the distant-dependent average received signal power with P_s being the transmit power, L a random variable corresponding to the large-scale fading power usually log-normal, and H is a random variable corresponding to the small-scale fading power. We assume that a non-singular path-loss model is used with

$$\ell(r) = C \max(d_0, r)^n, \quad (13)$$

where $n > 2$ is the path-loss exponent, $d_0 > 0$ is the reference distance, and $C > 0$ is a constant. In addition

to resulting in finite interference moments, (13) takes into account the realistic RF design constraints on the maximum received power and is often assumed in a standards based channel model like 3GPP LTE Advanced [38].

We consider the power due to small-scale fading as occurring after processing at the receiver, e.g. diversity combining. Furthermore, we assume that the transmit strategy is independent of the strategies in the interfering cells and no adaptive power control is employed. We model the small-scale fading power H as a Gamma distributed random variable with $\Gamma[k_h, \theta_h]$. The Gamma distribution allows us to model several different transmission techniques. For example, in conventional Rayleigh fading H is $\Gamma[1, 1]$ while Rayleigh fading with N_r receive antennas and maximum ratio combining H is $\Gamma[N_r, 1]$. Maximum ratio transmission with Rayleigh fading, N_t transmit antennas, and $N_r = 1$ receive antennas would have H is $\Gamma[N_t, 1]$. With multiuser MIMO (MU MIMO) with N_t transmit antennas, $U \leq N_t$ active users, and zero-forcing precoding then H is $\Gamma[N_t - U + 1, 1/U]$ where the $1/U$ follows from splitting the power among different users.

We model the large-scale fading contribution L as log-normal distributed with parameter σ^2 . The large-scale fading accounts for effects of shadowing due to large objects in the environment and essentially accounts for error in the fit of the log-distance path-loss model. Because of the difficulty in dealing with composite distributions, we approximate the random variable HL with another Gamma random variable with the same first and second order moments as described in Lemma 6 with distribution $\Gamma[k_p, \theta_p]$ where k_p is found in (8) and θ_p in (9). Note that from the scaling properties of Gamma random variables, with this approximation, the received signal power $S(r)$ is equivalently $\Gamma[k_p, P(r)\theta_p]$.

IV. HOMOGENEOUS NETWORK INTERFERENCE

In this section we develop a model for homogeneous network interference, i.e. interference comes from a single kind of interferer such as a macrocell. We model the interferer locations according to a marked point process, where the marks correspond to the channel between the interferers and the target receiver. Specifically we consider a PPP Φ with density λ and marks modeled similarly to the signal power distribution. For interferer k , L_k follows a log-normal distribution with σ while the small-scale fading distribution G_k is $\Gamma[k_g, \theta_g]$. The transmit power of the interferer is denoted by P_g . The transmit power is fixed for all interferers and the actions of each interferer are independent of the other interferers. Note that the parameters of the interference mark distribution are usually different than the signal channel even when the same transmission scheme is employed in all cells. For example, with single antenna transmission and conventional Rayleigh fading G_k is $\Gamma[1, 1]$ for any N_r at the receiver, while with multiuser MIMO zero-forcing beamforming with multiple transmit antennas serving $U \leq N_t$ users G_k is $\Gamma[U, 1]$.

We aim at computing the interference power received at a distance $0 < r < R$ away from the center of the cell as illustrated in Fig. 2. Unfortunately, the exclusion distance to

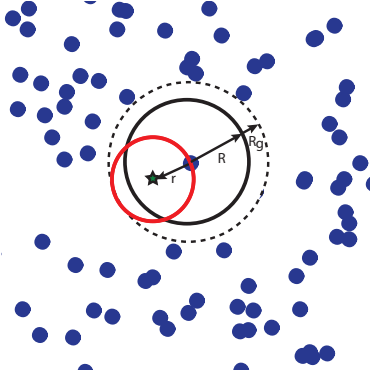


Fig. 2. Interference power calculation at a point r away from the transmitter. An upper bound is considered as the interference is not excluded uniformly around the receiver.

the nearest interferer is asymmetric, since the distance to the closest edge of the circle is $R_c + R_g - r$ while the distance to the furthest edge is $R_c + R_g + r$. To avoid the dependence on the location of the interference field we pursue an upper bound on the interference power, as illustrated Fig. 2. Specifically, we consider the interference contribution in a ball of radius $R_c + R_g - r$ around the received signal denoted $B(R_c + R_g - r)$. This results in an upper bound on the aggregate interference due to the fact that less area is excluded from the calculation. Then the receiver is at the center of the reduced interference region, and because it is a PPP thanks to Slivnyak's theorem [6] can be assumed to be at the origin. Under this assumption we write the received power as

$$I(r) = \sum_{k \in \Phi \setminus B(R_c + R_g - r)} \frac{G_k L_k P_g}{\ell(R_k)}. \quad (14)$$

We use the same non-singular path-loss model in (13) as for the transmit signal power. Other non-singular models and tradeoffs between different models are discussed in [8], [23]. As long as there is a guard region with $R_g > 0$ a singular path-loss model could be employed without changing the results. In our calculations we assume that $R_c + R_g - r > d_0$ to simplify the exposition.

We will characterize the distribution of the random variable $I(r)$ in terms of its Laplace transform, along the lines of [5]. When the interference marks $G_k L_k$ follow an arbitrary but identical distribution $\forall k$ the Laplace transform is given by

$$\begin{aligned} \mathcal{L}_{I(r)}(s) &= \mathbb{E}_{I(r)}[e^{-sI(r)}] \\ &= \mathbb{E}_{\Phi, G_k, L_k} \left(e^{-s \sum_{k \in \Phi \setminus B(R_c + R_g - r)} \frac{P_g G_k L_k}{\ell(R_k)}} \right) \\ &= \mathbb{E}_{\Phi, G_k, L_k} \left(\prod_{k \in \Phi \setminus B(R_c + R_g - r)} e^{-s \frac{P_g G_k L_k}{\ell(R_k)}} \right) \\ &\stackrel{(a)}{=} \mathbb{E}_{\Phi} \left(\prod_{k \in \Phi \setminus B(R_c + R_g - r)} \mathbb{E}_{G, L} \left(e^{-s \frac{P_g G L}{\ell(R_k)}} \right) \right) \\ &\stackrel{(b)}{=} e^{-2\pi\lambda \int_{R_c + R_g - r}^{\infty} (1 - \mathbb{E}_{G, L} (e^{-s P_g G L C^{-1} v^{-n}})) v dv} \end{aligned} \quad (15)$$

$$= e^{\lambda\pi(R_c + R_g - r)^2 - \frac{2\pi\lambda(sP_g)^{\frac{2}{n}}}{nC^{\frac{2}{n}}} F_{n, R_c + R_g - r, C}(s)}, \quad (16)$$

where (a) follows from the i.i.d. distribution of $G_k L_k$ and its further independence from the point process Φ , (b) follows assuming that $R_c + R_g - r > d_0$ and using the probability generating functional of the PPP [6], $Z = G_k L_k P_g$, and

$$F_{n, x, C}(s) = \mathbb{E}_Z z^{\frac{2}{n}} \left[\Gamma\left(-\frac{2}{n}, sC^{-1}zx^{-n}\right) - \Gamma\left(-\frac{2}{n}\right) \right]. \quad (17)$$

In some cases the expectation can be further evaluated, e.g. when $Z \sim \Gamma[k, \theta]$ then $F_{n, x, C}(s) = \frac{(sC^{-1}x^{-n})^{-\frac{2}{n}-k} n\theta^{-k}}{2+kn} {}_2F_1\left(k, k + \frac{2}{n}, 1 + k + \frac{2}{n}, -\frac{1}{sC^{-1}x^{-n}\theta}\right) - \theta^{2/n} \mathcal{B}\left(k + \frac{2}{n}, -\frac{2}{n}\right)$ where $\mathcal{B}(x, y)$ is the Beta Euler function and ${}_2F_1$ is the Gauss hypergeometric function.

V. HETEROGENEOUS NETWORK INTERFERENCE

In this section we extend the proposed interference model to the case of a heterogeneous network, where the macrocellular network is underlaid with other kinds of infrastructure including low-power nodes like small-cells, femtocells, fixed relays, and distributed antennas. The key idea is to model the interference from each type of infrastructure separately using a marked PPP described by the model in Section IV, and then develop an equivalent marked PPP for the sum interference term to facilitate analysis. We consider two different interference scenarios as illustrated in Fig. 3. The scenario in Fig. 3(a) corresponds to multiple kinds of out-of-cell interference. For example, there might be interference from both high power base stations and low power distributed antenna system transmission points. The guard regions associated with different processes may be different. The scenario in Fig. 3(b) corresponds to cross-tier interference from a second tier of low-power nodes, e.g. small-cells or femtocells or uncoordinated transmission points. The main difference in this case is that there is interference within the cell so effectively the interference contribution is a constant since it is spatially invariant. There may still be a guard radius in this case but it is likely to be small.

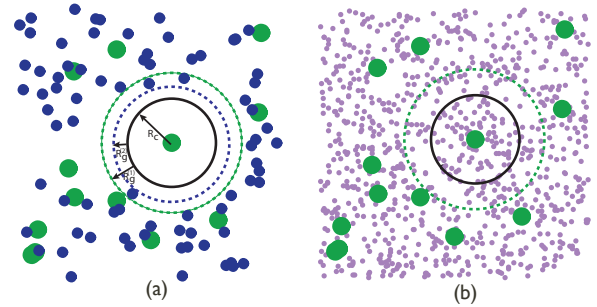


Fig. 3. Interference in a heterogeneous network from multiple kinds of infrastructure. (a) Heterogeneous out-of-cell interference. (b) Homogeneous interference and cross-tier interference from low power nodes.

A. Heterogeneous Out-of-Cell Interference

Suppose that there are M different kinds of infrastructure deployed in a homogenous way through the network. A more complex model would take non-uniformity or clustering into account, but this is beyond the scope of this work. Each interference source can be modeled as a marked PPP with marks corresponding to the composite fading distribution distributed as $\Gamma[k_m, \theta_m]$ where $m = 1, \dots, M$. The path-loss function is the same for each process. The transmitting power is given by P_m , the transmitter density by λ_m , and the guard radius by $R_g^{(m)}$. Consequently, each process is parameterized by $(k_m, \theta_m, P_m, \lambda_m, R_g^{(m)})$. This gives an equation for the total interference as

$$J_{\text{tot}}(r) = \sum_{m=1}^M \sum_{k \in \Phi_m \setminus B(R_c + R_g^{(m)} - r)} \frac{G_k^{(m)} L_k^{(m)} P_m}{\ell(R_k^{(m)})}. \quad (18)$$

The difference in guard regions adds a complication to developing an equivalent marked PPP. Consequently we upperbound the interference level as described in the following lemma.

Lemma 7: Consider the sum of m PPPs Φ_m with parameters $(k_m, \theta_m, P_m, \lambda_m, R_g^{(m)})$. Then

$$J_{\text{tot}}(r) \leq \sum_{m=1}^M \underbrace{\sum_{k \in \Phi_m \setminus B(R_c + R_{\min} - r)} \frac{G_k^{(m)} L_k^{(m)} P_m}{\ell(R_k^{(m)})}}_{J^{(m)}(r)} := J(r) \quad (19)$$

where $R_{\min} = \min_{1 \leq m \leq M} R_g^{(m)}$ and $X \leq^{\text{st}} Y$ denotes that X is stochastically dominated by Y .

Proof: Follows directly by noting that $\text{vol}(B(R_c + R_g^{(m)} - r)) \geq \text{vol}(B(R_c + R_{\min} - r))$ and that $\Phi_m \setminus B(R_c + R_g^{(m)} - r) \subset \Phi_m \setminus B(R_c + R_{\min} - r)$. ■

The superposition of M different homogeneous PPPs can be treated as a single equivalent PPP by modifying appropriately the mark distribution. This many simplify analysis and simulations in some cases.

Proposition 8: The cumulative interference received from all M interfering networks

$$\sum_{m=1}^M \sum_{k \in \Phi_m \setminus B(R_c + R_{\min} - r)} \frac{G_k^{(m)} L_k^{(m)} P_m}{\ell(R_k^{(m)})} = \sum_{k \in \Phi_e} \frac{X_k^{(e)}}{\ell(R_k^{(e)})}, \quad (20)$$

where $\Phi_e = \left\{ \bigcup_{m=1}^M \Phi_m \setminus B(R_c + R_{\min} - r) \right\}$ and $X_k^{(e)}$ is a random variable that is equal to $G^{(m)} L^{(m)} P_m$ with probability $\lambda_m / \sum_{m=1}^M \lambda_m$ for $m = 1, \dots, M$.

Proof: The union of M PPPs, i.e. $\Phi_s = \left\{ \bigcup_{m=1}^M \Phi_m \setminus B(R_c + R_{\min} - r) \right\}$, is a PPP with intensity measure $\lambda_e = \sum_{m=1}^M \lambda_m$ since the superposition of M independent PPPs is also a PPP [8]. Consider now a randomly

chosen point (typical point) of $x \in \Phi_s$, then the probability for the event that x belongs to $\Phi_m \setminus B(R_c + R_{\min} - r)$, i.e. $\mathbb{P}(x \in \Phi_m \setminus B(R_c + R_{\min} - r)) = \frac{\lambda_m}{\sum_{m=1}^M \lambda_m}$. Hence, the random variable $X_k^{(e)}$ (interference mark) of any point $x \in \Phi_s$ is equal to $G^{(m)} L^{(m)} P_m$ with probability $\mathbb{P}(x \in \Phi_m \setminus B(R_c + R_{\min} - r)) = \lambda_m / \sum_{m=1}^M \lambda_m$. ■

The marks of the cumulative heterogeneous interference has a finite mixture distribution $f_X(x)$ given by

$$f_X(x) = \sum_m \frac{\lambda_m}{\sum_{m=1}^M \lambda_m} f_Z^{(m)}(z), \quad (21)$$

where $f_Z^{(m)}(z)$ denotes the PDF of $Z^{(m)} = G^{(m)} L^{(m)} P_m$.

From the independence of the M PPPs and using similar arguments as for deriving (16), the Laplace transform of the heterogeneous out-of-cell interference is

$$\mathcal{L}_J(s) = \prod_{m=1}^M \mathcal{L}_{J^{(m)}}(s), \quad (22)$$

with $\mathcal{L}_{J^{(m)}}(s)$ the Laplace transform of $J^{(m)}(r)$ defined as in Lemma 7 with appropriate substitution for P_m , λ_m , and R_{\min} .

Expanding the integrals in (22) along the lines of the calculations that lead to (16), combining terms, and using the equivalent mark distribution from Proposition 8 leads to the following result.

Corollary 9: The Laplace transform of the cumulative interference from all M tiers is given by

$$\mathcal{L}_J(s) = e^{\lambda_e \pi (R_c + R_{\min} - r)^2 - \frac{2\pi \lambda_e (sC^{-1})^{\frac{2}{n}}}{n} F_{n, R_c + R_{\min} - r, C}(s)} \quad (23)$$

where $\lambda_e = \sum_{m=1}^M \lambda_m$.

Using Corollary 9 we can replace calculations involving the heterogeneous interference term with an equivalent marked interference process. Consequently the homogeneous interference results can be applied but with a different mark distribution and a total density term.

B. Cross-Tier Interference

Certain kinds of infrastructure, like femtocells, are not associated with the macro base station. Rather they form another tier of nodes and create cross-tier interference. We propose to use the same stochastic geometry framework to model the interference with a main difference in how the notion of guard zone is defined. Let B_{cross} denote an exclusion region around the desired receiver with radius R_{cross} . This region might be quite small, just a few meters radius, and is designed to avoid the case where the low-power node is co-located with the target receiver. Let us suppose that the cross-tier interference is associated with a PPP given by Φ_{cross} , the transmitting power is given by P_{cross} , and the transmitter density by λ_{cross} . The received interference power is given by

$$I_{\text{cross}} = \sum_{k \in \Phi_{\text{cross}} \setminus B_{\text{cross}}} \frac{G_k L_k P_{\text{cross}}}{\ell(R_k)}, \quad (24)$$

which does not depend on r due to the shift invariance property of the PPP. Consequently cross-tier interference creates a constant interference that is independent of the mobile receiver location.

VI. APPROXIMATING THE INTERFERENCE DISTRIBUTION USING THE GAMMA DISTRIBUTION

The expressions in Section IV and Section V provide a characterization of the distribution of the interference terms in the proposed hybrid approach. Unfortunately, the distributions are characterized in terms of the Laplace transform and further simplifications exist only for special choices of the parameters and mark distribution. Consequently, in this section, we pursue an approximation of the distribution of the interference using the Gamma distribution. First we review the calculations for the Gamma approximation of the interference terms with a single and multiple interferers. Then we use these calculations to parameterize the Gamma approximation of the interference.

A. Moments of the Interference Process

The moments for interference distribution can be computed along the lines of [8, Equations 2.19 and 2.21]. We only need the first two moments for the Gamma approximation. Let $Z = G_k L_k P_g$ denote the random variable corresponding to the mark distribution for the case of a homogeneous interference source. Then using results from [8]

$$\mathbb{E}I(r) = (\mathbb{E}Z) 2\pi\lambda \int_{R_c+R_g-r}^{\infty} \frac{w}{\ell(w)} dw \quad (25)$$

$$\mathbb{E}I(r)^2 = (\mathbb{E}I(r))^2 + (\mathbb{E}Z^2) 2\pi\lambda \int_{R_c+R_g-r}^{\infty} \frac{w}{\ell(w)^2} dw. \quad (26)$$

The interference is a type of shot noise. For the nonsingular model of (13), the value is finite for any r . Now we assume that $R_c + R_g - r > d_0$ to simplify the exposition. We summarize the results in the following proposition.

Proposition 10 (Moments of Homogeneous Interference): The mean and variance of the interference in the case of homogenous interference are

$$\mathbb{E}I(r) = (\mathbb{E}Z) 2\pi\lambda \frac{(R_c + R_g - r)^{2-n}}{C(n-2)} \quad (27)$$

$$\text{var } I(r) = (\mathbb{E}Z^2) 2\pi\lambda \frac{(R_c + R_g - r)^{2-2n}}{2C^2(n-1)}. \quad (28)$$

In the case of heterogeneous interference, we propose to use the moments of the sum to approximate using the equivalent mark distribution. We summarize the key results in the following proposition.

Proposition 11 (Moments of Heterogeneous Interference): The mean and variance of the interference in the case of heterogeneous interference are

$$\mathbb{E}J(r) = \sum_{m=1}^M \left(\mathbb{E}Z^{(m)} \right) 2\pi\lambda_m \frac{(R_c + R_g^{(m)} - r)^{2-n}}{C(n-2)} \quad (29)$$

$$\text{var } J(r) = \sum_{m=1}^M \left(\mathbb{E}X^{(m)2} \right) 2\pi\lambda_m \frac{(R_c + R_g^{(m)} - r)^{2-2n}}{2C^2(n-1)} \quad (30)$$

where $Z^{(m)} = G_k^{(m)} L_k^{(m)} P_m$.

The calculations can be further simplified using the minimum radius approximation but we defer their presentation due to lack of space.

The approach in (11) is quite general. An additional term could be included accounting for the cross-tier interference I_{cross} in a straightforward way. Further modifications are possible. For example, having different numbers of active users in an interfering cell with multiuser MIMO could be included by having a sum of marked point processes each with a different fading distribution corresponding to different choices of active users. Note that we can incorporate cross-tier interference I_{cross} into Proposition 11 by replacing $R_c + R_g - r$ with R_{cross} and selecting the other parameters as appropriate.

B. Gamma Approximation of the Interference

While the Laplace transform of the interference power completely characterizes the distribution, it is often challenging to provide simple or closed-form solutions for a variety of scenarios of interest. Consequently we pursue an approximation of the interference term that will yield simpler expressions targeted at the proposed hybrid setting using the Gamma distribution.

Proposition 12 (Homogeneous Interference): The $\Gamma[k_i, \theta_i]$ random variable with the same mean and variance as $I(r)$ has

$$k_i = 2\pi\lambda(R_c + R_g - r)^2 \frac{(\mathbb{E}Z)^2}{\mathbb{E}Z^2} \frac{2(n-1)}{(n-2)^2} \quad (31)$$

and

$$\theta_i = \frac{\mathbb{E}Z^2}{\mathbb{E}Z} (R_c + R_g - r)^{-n} \frac{1}{2C} \frac{n-2}{n-1}. \quad (32)$$

Proof: Using the results of Lemma 3 with $\mathbb{E}I(r) = k_i \theta_i$ and $\text{var } I(r) = k_i \theta_i^2$, then substituting (27) and (28), gives k_i and θ_i . ■

We can derive a similar expression in the case of heterogeneous interference, using Lemma 3. We omit the result due to lack of space.

Now suppose that the mark distribution $P_g L_k G_k$ is approximated by a Gamma distribution. Substituting and simplifying leads to the following corollary.

Corollary 13 (Homogeneous Interf. – Gamma Fading): Suppose that the product fading distribution $L_k G_k$ is $\Gamma[k_z, \theta_z]$. Then the parameters of the second order Gamma approximation are

$$k_i = \frac{k_z}{1+k_z} 4\pi\lambda(R_c + R_g - r)^2 \frac{(n-1)}{(n-2)^2} \quad (33)$$

and

$$\theta_i = P_g \theta_z (1+k_z) (R_c + R_g - r)^{-n} \frac{1}{2C} \frac{n-2}{n-1}. \quad (34)$$

Proof: If Z is $P_g L_k G_k$ where $L_k G_k$ is $\Gamma[k_z, \theta_z]$ then $\mathbb{E}Z = k_z P_g \theta_z$ and $\mathbb{E}Z^2 = k_z P_g^2 \theta_z^2 (1+k_z)$. Substituting into (31) and (32) gives the result. ■

We also provide the result for the heterogeneous case where each tier of interferers has power P_m and product fading distribution given by $G_k^{(m)} L_k^{(m)}$ with distribution $\Gamma[k_m, \theta_m P_m]$. Substituting and simplifying gives the second corollary.

Corollary 14 (Heterogeneous Interf. – Gamma Fading): Suppose there are M tiers of interference, each with power P_m and product fading distribution given by $G_k^{(m)} L_k^{(m)}$ approximated as $\Gamma[k_m, \theta_m P_m]$. Then the parameters of the second order Gamma approximation are

$$k_i = \frac{(\mathbb{E}J(r))^2}{\text{var}J(r)} \quad \text{and} \quad \theta_i = \frac{\text{var}J(r)}{\mathbb{E}J(r)} \quad (35)$$

where

$$\begin{aligned} \mathbb{E}J(r) &= \sum_{m=1}^M k_m \theta_m P_m 2\pi \lambda_m \frac{(R_c + R_g^{(m)} - r)^{2-n}}{C(n-2)} \\ \text{var} J(r) &= \sum_{m=1}^M k_m \theta_m^2 (1 + k_m) 2\pi \lambda_m \frac{(R_c + R_g^{(m)} - r)^{2-2n}}{2C^2(n-1)}. \end{aligned} \quad (36) \quad (37)$$

Proof: The main equation follows from Lemma 3, inserting the moments for each interference tier computed along the lines of Proposition 10. ■

VII. CHARACTERIZATION OF THE RANDOM SIR

In this section we characterize the system performance as a function of the random quantity the signal to interference ratio (SIR) given by

$$\text{SIR}(r) = \frac{S(r)}{J(r)}. \quad (38)$$

The SIR is a useful quantify for performance analysis in cellular systems because performance at the cell edge is usually interference limited; we consider additive noise and signal to interference plus noise ratio (SINR) in the simulations. Because simplified expressions including the exact distribution calculated in Section IV-Section V are only available in special cases, we use the Gamma approximation described in Section VI. To make the results concrete, we assume that $S(r) = S_r$ where $S_r \sim \Gamma[k_s, \theta_s]$. Note that in the distant-dependent path-loss and transmit power are absorbed into the θ_s term for ease of presentation.

We use the $\text{SIR}(r)$ to evaluate two metrics of performance: the success probability defined as $\mathbb{P}(\text{SIR}(r) > T)$ where T is some threshold and the ergodic achievable rate given by

$$\tau(r) := \mathbb{E} \ln(1 + \text{SIR}(r)). \quad (39)$$

The success probability is one minus the outage probability and gives a measure of diversity performance and severity of the fading. The achievable rate gives the average rate that can be achieved at r assuming that the interference is Gaussian.

A. Success Probability

We first provide the success probability without proceeding to a Gamma approximation for the out-of-cell interference and signal distribution given by $S(r) = S_r$ where $S_r \sim \Gamma[k_s, \theta_s]$. Using the results from [41], the success probability at location r is given by

$$\mathbb{P}(\text{SIR}(r) > T) = \sum_{j=0}^{k_s-1} \frac{1}{j!} \left(-\frac{\ell(r)T}{P_s \theta_s} \right)^j \frac{d^j \mathcal{L}_I(s)}{ds^j} \Big|_{s=\frac{\ell(r)T}{P_s \theta_s}} \quad (40)$$

where in the case of heterogeneous interference, $\mathcal{L}_I(s)$ is given by (16). In the special case of $k_s = 1$ the success probability can be simplified as

$$\mathbb{P}(\text{SIR}(r) > T) = \mathcal{L}_I(\ell(r)T/(P_s \theta_s)). \quad (41)$$

Evidently, the closed-form expression (40) is involved and no useful insight can be obtained. This motivates the use of Gamma approximation for the aggregate interference as a means to provide simpler and useful expressions.

We evaluate below the success probability for the case of Gamma distributed interference, effectively $J(r) \sim \Gamma[k_i, \theta_i]$. In the case of homogeneous interference, the expression for k_i is found in (31) and for θ_i in (32). In the case of heterogeneous interference, the expressions for k_i and θ_i are found in (35). The result is summarized in the following proposition.

Proposition 15 (Success Prob. for Gamma Interference): The success probability when the signal distribution is $S(r) = S_r$ where $S_r \sim \Gamma[k_s, \theta_s]$ and the interference distribution is $J(r) = I_r$ where $I_r \sim \Gamma[k_i, \theta_i]$ is given by

$$\begin{aligned} \mathbb{P}(\text{SIR}(r) > T) &= \frac{\Gamma(k_s + k_i)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_i} \right)^{k_i} \\ &\times {}_2\tilde{F}_1 \left(k_i, k_s + k_i, 1 + k_i, -\frac{\theta_s}{T\theta_i} \right) \end{aligned} \quad (42)$$

where ${}_2\tilde{F}_1$ is a regularized hypergeometric function.

Proof: First we rewrite the probability to separate the signal and interference terms as $\mathbb{P}(\text{SIR}(r) > T) = \mathbb{P}(S_r > T I_r)$. Now conditioning on the interference and evaluating the expectation $\mathbb{P}(\text{SIR}(r) > T) = \mathbb{E}(F_S(T I_r)) =$

$$\frac{\Gamma(k_s + k_i)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_i} \right)^{k_i} {}_2\tilde{F}_1 \left(k_i, k_s + k_i, 1 + k_i, -\frac{\theta_s}{T\theta_i} \right) \quad (43)$$

where $F_S(\cdot)$ is the Gamma cumulative distribution function of S_r and the second equality follows from evaluating the expectation. ■

We can obtain a more exact expression in the case where each interferer is separately assumed to have a Gamma distribution.

Proposition 16 (Heterogeneous Success Probability): The success probability when the signal distribution is $S(r) = S_r$ where $S_r \sim \Gamma[k_s, \theta_s]$ and the interference distribution is $J(r) = J_r^{(1)} + J_r^{(2)} + \dots + J_r^{(M)}$ where $J_r^{(m)} \sim \Gamma[k_m, \theta_m]$ for $m = 1, 2, \dots, M$ is given by

$$\begin{aligned} \mathbb{P}(\text{SIR}(r) > T) &= C \sum_{n=0}^{\infty} c_n \frac{\Gamma(k_s + \rho + n)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_{\min}} \right)^{n+\rho} \\ &\times {}_2\tilde{F}_1 \left(n + \rho, k_s + n + \rho, 1 + n + \rho, -\frac{\theta_s}{T\theta_{\min}} \right) \end{aligned} \quad (44)$$

where the parameters C, ρ, θ_{\min} are found using the results of Theorem 4.

Proof: First we rewrite the probability to separate the signal and interference terms, and then conditioning on the interference, we evaluate the expectation, i.e.

$$\mathbb{P}(\text{SIR}(r) > T) = \mathbb{P} \left(S_r > T \sum_m I_r^{(m)} \right) = \mathbb{E} \left(F_S \left(T \sum_m I_r^{(m)} \right) \right). \quad (45)$$

From the results of Theorem 4, the distribution of $\sum_m I_r^{(m)}$ can be expressed as

$$f_Y(y) = C \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(\rho+n)\theta_{\min}^{\rho+n}} y^{\rho+n-1} e^{-y/\theta_{\min}} \quad (46)$$

where $\theta_{\min} := \min_m \theta_i^{(m)}$, $\rho = \sum_{m=1}^M k_i^{(m)}$, $C = \prod_{m=1}^M (\theta_{\min}/\theta_i^{(m)})^{k_i^{(m)}}$, and c_m is found by recursion from

$$c_{n+1} = \frac{1}{n+1} \sum_{p=1}^{n+1} p \gamma_p c_{n+1-p} \quad (47)$$

with $\gamma_p = \sum_{m=1}^M k_i^{(m)} p^{-1} (1 - \theta_{\min}/\theta_i^{(m)})^n$. Now recognize that (5) is essentially a mixture of $\Gamma[\rho+n, \theta_{\min}]$ random variables with weights $C c_n$. Now we can use the results of (15) to establish that

$$\begin{aligned} \mathbb{P}(\text{SIR}(r) > T) &= C \sum_{n=0}^{\infty} c_n \frac{\Gamma(k_s + \rho + n)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_{\min}} \right)^{n+\rho} \\ &\times {}_2\tilde{F}_1 \left(n + \rho, k_s + n + \rho; 1 + n + \rho; -\frac{\theta_s}{T\theta_{\min}} \right). \end{aligned} \quad (48)$$

While the result in Proposition 16 is more exact, it requires truncating the infinite series. In related work [37] we found that in most cases the approximation outperforms the truncated expression in (17) unless a large number of terms are kept, and therefore conclude that the expression in Proposition 16 is most convenient.

The regularized hypergeometric function ${}_2\tilde{F}_1(a, b, c, z)$ is available in many numerical packages or can be computed from unregularized hypergeometric function by recognizing that ${}_2\tilde{F}_1(a, b, c, z) = \frac{{}_2F_1(a, b, c, z)}{\Gamma(c)}$. For small values of $|z|$ it is convenient to compute it using the truncated series definition [42], [43] while for larger values of $|z|$ the asymptotic expression can be employed [44]. Depending on which numerical package is employed, it may be faster to compute the SIR using Monte Carlo techniques based on $\Gamma[k_s, \theta_s]$ and $\Gamma[k_i, \theta_i]$.

B. Average Rate

The average rate is useful for computing average rates at the cell edge and the area spectral efficiency, where the average rate at r is integrated over the radius of the cell. Our calculations are based on the observation that $\mathbb{E} \ln(1 + \text{SIR}(r)) = \mathbb{E} \ln(S_r + I_r) - \mathbb{E} \ln(I_r)$. From Lemma 2, $\mathbb{E} \ln(I_r)$ has a computable solution but $\mathbb{E} \ln(S_r + I_r)$ requires dealing with a sum of Gamma random variables. To solve this problem, we can use the result in (5) to find an expression for the expected log of the sum of Gamma random variables. The resulting expression is summarized in the following proposition.

Proposition 17: Suppose that $\{X_m\}_{m=1}^M$ are independent $\Gamma[k_m, \theta_m]$ distributed random variables. Then

$$\mathbb{E} \ln \left(\sum_{m=1}^M X_m \right) = \psi(\rho) + \ln(\theta_{\min}) + C \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n-1} \frac{1}{\rho + i}. \quad (49)$$

Proof: See [37, Proposition 5]. ■

Using this result, the average rate with homogeneous interference follows.

Proposition 18 (Homogenous Average Rate): The average rate with homogeneous interference in the absence of noise is $\tau(r) = \mathbb{E} \ln(1 + \text{SIR}(r)) =$

$$C \sum_{n=0}^{\infty} c_n \ln(\theta_{\min}) + C \sum_{n=0}^{\infty} c_n \psi(\rho + n) - \psi(k_i) - \ln(\theta_i) \quad (50)$$

where $\theta_{\min} := \min\{\theta_s, \theta_i\}$, $\rho = k_s + k_i$, $C = (\theta_{\min}/\theta_s)^{k_s} (\theta_{\min}/\theta_i)^{k_i}$, and c_n is found by recursion in (6).

Proof: Follows from the application of Proposition 17 to the case of the sum of two Gamma random variables $\Gamma[k_s, \theta_s]$ and $\Gamma[k_i, \theta_i]$, and using the result in Lemma 2. ■

The average rate with heterogeneous interference can be computed along the same lines as Proposition 18.

Proposition 19 (Heterogeneous Average Rate): The average rate with homogeneous interference in the absence of noise is

$$\begin{aligned} \tau(r) &= C \sum_{n=0}^{\infty} c_n \ln(\theta_{\min}^{(si)}) + C \sum_{n=0}^{\infty} c_n \psi(\rho^{(si)} + n) \\ &- D \sum_{n=0}^{\infty} d_n \ln(\theta_{\min}^{(i)}) - D \sum_{n=0}^{\infty} d_n \psi(\rho^{(i)} + n) \end{aligned} \quad (51)$$

where $\theta_{\min}^{(si)} := \min\{\theta_s, \theta_i^{(1)}, \theta_i^{(2)}, \dots, \theta_i^{(M)}\}$, $\rho^{(si)} = k_s + k_i^{(1)} + \dots + k_i^{(M)}$, $\theta_{\min}^{(i)} := \min\{\theta_i^{(1)}, \theta_i^{(2)}, \dots, \theta_i^{(M)}\}$, $C = (\theta_{\min}/\theta_s)^{k_s} \prod_{m=1}^M (\theta_{\min}/\theta_i^{(m)})^{k_i^{(m)}}$, $\rho^{(i)} = k_i^{(1)} + \dots + k_i^{(M)}$, $D = \prod_{m=1}^M (\theta_{\min}/\theta_i^{(m)})^{k_i^{(m)}}$, and c_m and d_m are found by recursions like in (6).

Proof: The first pair of terms result from application of Proposition 17 to the term $\mathbb{E} \ln S_r + I_r^{(1)} + I_r^{(2)} + \dots + I_r^{(M)}$, while the second term results from application of Proposition 17 to the term $\mathbb{E} \ln I_r^{(1)} + I_r^{(2)} \dots I_r^{(M)}$. ■

If the number of terms is too large for practical computation, an alternative is to approximate the sum of the signal plus interference term as another Gamma random variable using the moment matching approach.

Proposition 20 (Homogeneous Average Rate – Gamma): The average achievable rate without noise with homogeneous interference is approximately

$$\begin{aligned} \tau(r) &\approx \psi \left(\frac{(k_s \theta_s + k_z \theta_z)^2}{k_s \theta_s^2 + k_z \theta_z^2} \right) + \ln \left(\frac{k_s \theta_s^2 + k_z \theta_z^2}{k_s \theta_s + k_z \theta_z} \right) \\ &- \psi(k_i) - \ln(\theta_i) \end{aligned} \quad (52)$$

Proof: To find $\mathbb{E} \ln(S_r + I_r)$ we apply Lemma 5 and approximate the sum of Gamma random variables with another Gamma distribution with the same mean and variance. Note that

$$\mathbb{E} S_r + I_r = k_s \theta_s + k_i \theta_i \quad (53)$$

$$\text{var } S_r + I_r = k_s \theta_s^2 + k_i \theta_i^2. \quad (54)$$

Then a Gamma random variable $Z \sim \Gamma[k_z, \theta_z]$ with the same mean and variance has

$$k_z = \frac{(k_s \theta_s + k_z \theta_z)^2}{k_s \theta_s^2 + k_z \theta_z^2} \quad \theta_z = \frac{k_s \theta_s^2 + k_z \theta_z^2}{k_s \theta_s + k_z \theta_z}. \quad (55)$$

Now we can approximate the term

$$\mathbb{E} \ln(S_r + I_r) \approx \psi \left(\frac{(k_s \theta_s + k_z \theta_z)^2}{k_s \theta_s^2 + k_z \theta_z^2} \right) + \ln \left(\frac{k_s \theta_s^2 + k_z \theta_z^2}{k_s \theta_s + k_z \theta_z} \right). \quad (56)$$

The last part of the expression for $\mathbb{E} \ln I_r$ follows from Lemma 2. ■

The result can be extended to the case of heterogeneous interference by modifying the computation of the mean and variance terms.

VIII. SIMULATIONS

In this section we consider a system model of the form in Fig. 2 with a single antenna base station and a Poisson field of base station interferers. We evaluate performance in a cell of radius $R_c = 300m$ with a guard region of $R_g = 150m$. The base station locations are modeled as a PPP with $\lambda = 2/\sqrt{27}R_c^2 = 4.3 \cdot 10^{-6}$ transmitters per m^2 which corresponds to the same density as a hexagonal lattice with cells of radius R_c . The base station has 40W transmit power with a single antenna. Independent identically distributed complex Gaussian fading is assumed (corresponds to Rayleigh distributed channel amplitudes). The path-loss model of (13) is used with $n = 3.76$, $d_0 = 35m$, and $C = 33.88$. Log-normal shadowing is assumed with 6dB variance. Monte Carlo simulations are performed by simulating base station locations over a square of dimension $15R_c \times 15R_c$. The mutual information and the outage probability are estimated from their sample averages over 200 small scale fading realizations, and 500 different interferer positions. A probability of success threshold of $T = 31$ (corresponding to 5dB) is used. Performance is evaluated as a function of the distance from the center of the fixed cell.

First we validate two of the key assumptions used in most of the results in this paper: the use of a guard region and the approximation that the interference power for a user at radius r from the cell center is upper bounded by considering interferers outside a ball of radius $R_c + R_g - r$. In Fig. 4 we compare the performance between the proposed hybrid approach and a hexagonal layout. We consider three different guard regions for the hybrid approach: $R_g = 0$, $R_g = R_c$, and $R_g = 2R_c/3$. We compute the received interference power for a user at radius r from the cell center by considering that the interference lies outside a ball of radius $R_c + R_g - r$ as illustrated in Fig. 2. We compare with a hexagonal layout of cells with either one tier of interferers (six total) or two tiers of interferers (eighteen total). We plot the ergodic rate in Fig. 4 and the probability of success in Fig. 5. Our approach with a guard region of $R_g = 2R_c/3$ matches the multi-tier hexagonal layout both in terms of ergodic rate and probability of success. Consequently our numerical results justify our claim that a Poisson interference field can approximate an interference field from base stations located on a hexagonal grid if a suitably chosen guard region is employed.

Now we consider the Gamma approximation for the interference field. We claim that the Gamma distribution is a good approximation of the distribution strength of the interference

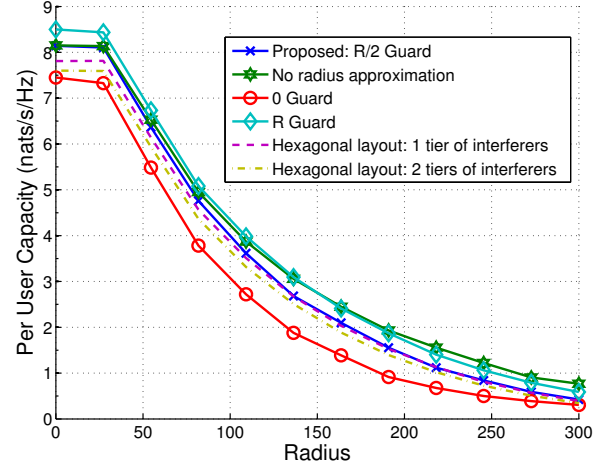


Fig. 4. Illustrates the ergodic rate as a function of radius from the cell center. The flat part at the beginning is due to the non-singular path-loss model that is constant until the reference distance of d_0 . The case of no guard interval provides a lower bound on performance while a guard interval of R_c provides an upper bound on performance. Performance without the interference field approximation is good near the cell center but deviates further away from the base station. The hexagonal layout with one and two tiers of interference almost exactly overlaps the performance with a guard of $2R_c/3$. This justifies both the symmetric interference field approximation and the use of a guard interval $R_g = 2R_c/3$. Note that in this case we compare the Poisson distributed interferers with the hexagonal layout of interferers to justify our claim, there is no Gamma approximation of the interference.

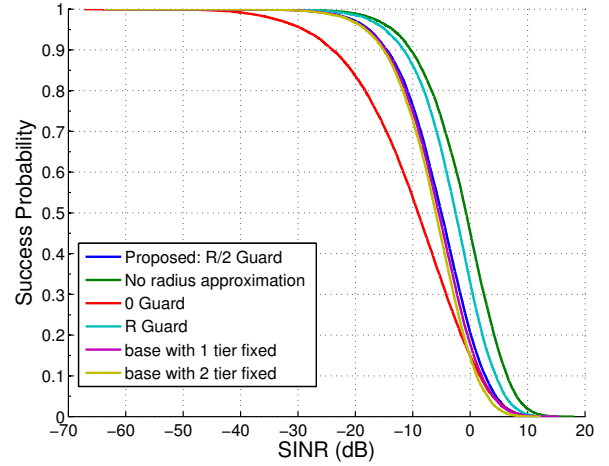


Fig. 5. Probability of success curves derived from the Kaplan-Meier estimate of the cumulative distribution function at a distance of $r = 300m$. Due to the number of points this curve is best viewed in color. Similar results are observed as for the average rate. The case of no guard interval provides a lower bound on performance while a guard interval of R_c provides an upper bound on performance. Performance without the interference field approximation is good near the cell center but deviates further away from the base station. The hexagonal layout with one and two tiers of interference almost exactly overlap the performance with a guard of $2R_c/3$. This further justifies both the symmetric interference field approximation and the use of a guard interval $R_g = 2R_c/3$. Note that in this case we compare the Poisson distributed interferers with the hexagonal layout of interferers, there is no Gamma approximation of the interference.

power. As an error metric we use the percent error in the expected log as given by

$$\text{error} = \frac{|\mathbb{E} \ln X_{\text{gamma}} - \mathbb{E} \ln I(r)|}{|\mathbb{E} \ln I(r)|}.$$

The motivations for comparing $\mathbb{E} \ln X$ are that it shows up in the calculation of the average rate and that signal and interference power are usually measured on a log scale. Note that we match moments on the linear scale (not the log scale) thus the comparison seems to be fair. The comparison is described in detail in Fig. 6. Based on our comparison we conclude that with a suitably chosen guard region of $2R_c/3$, the gamma approximation gives an error in the log that is acceptable, justifying our use of the Gamma approximation of the interference field.

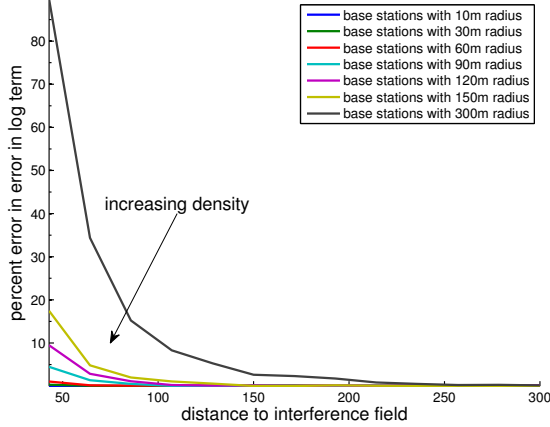


Fig. 6. In this simulation, the intensity of the interference field is varied according to $\lambda = 2/\sqrt{27}R_i^2$ by changing the size of R_i . There is no guard region. It can be seen that the approximation is better for higher intensity interference. For our simulations with $R_c = 300m$ we see that the percent error at the location of the guard region $2R_c/3 = 200m$ is less than 3%.

Now we compare the Gamma approximation in terms of the average rate and the success probability in a case with heterogeneous interference. We consider a setup with a single antenna base station and three sources of interference. One source of interference is from other base stations, which has power and density as we already described. The second source of interference is a multiuser MIMO distributed antenna system (DAS) with four single antenna transmission points, a single active user per cell, and the same total power. The DAS system is also modeled using a PPP but with density $4\lambda = 1.7 \cdot 10^{-5}$. The distribution of the effective channels are derived from [37]. Note that we assume essentially that every cell also has a DAS system, in reality some cells would have DAS and some would not, which can be included by modifying the density. The third source of interference is from cross-tier interference from a single tier of femtocells. The femto interferers have a $20dBm$ transmit power and a guard region of $5m$. Since this is cross-tier interference, the femtocells may be deployed in the cell and thus they create a constant level of interference power corresponding to a minimum distance of $5m$. An indoor-outdoor penetration loss of $20dB$ is assumed. The results of the comparison in this section are papered in Fig. 7 for average capacity and in Fig. 8 for outage. In terms of average rate, the main conclusion is that the Gamma approximation provides good performance for heterogeneous interference. There is a more significant error at the cell edge for only a single interferer. In terms of success

probability, similar results are observed. There is some error with the homogeneous interference at the cell edge, the fit with homogeneous and DAS interference is good, and the inclusion of femtocells reduces the accuracy slightly. While there is approximation error, we still believe that the fit is good enough to justify the viability of the Gamma distribution.

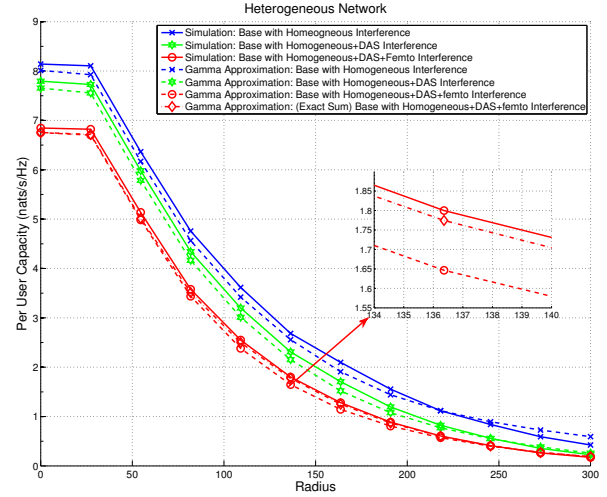


Fig. 7. Comparison of the ergodic rates with and without the Gamma approximation with a tier of low-power femto nodes that create cross-tier interference. We see that the Gamma approximation provides low error except towards the cell edge of homogeneous interference. With cross-tier interference, the Gamma approximation provides a constant offset of the entire simulation curve. The offset is constant because the cross-tier interference is constant. The error though is consistent, meaning that it is still possible to make relative comparisons between the Gamma approximation curves.

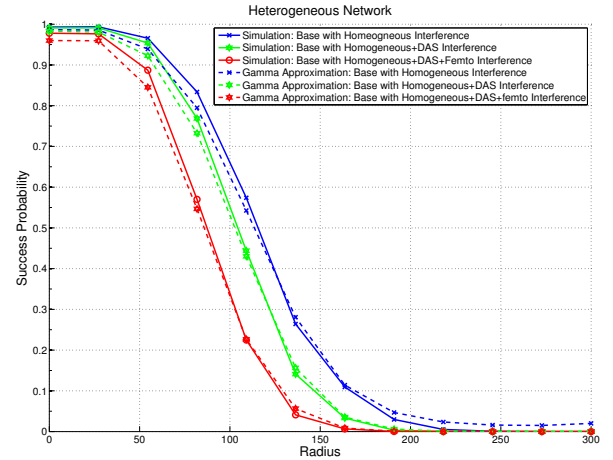


Fig. 8. Comparison of the success probability with and without the Gamma approximation for a target SIR of $15dB$. We see that the Gamma approximation overall provides reasonable performance except right at the cell edge. At small SIR it tends to underestimate the outage. There is about a 15% error with small cell interference versus having homogeneous interference. The outage curves were computed using Monte Carlo techniques.

IX. CONCLUSIONS

In this paper, we proposed a hybrid model for determining the impact of interference in cellular systems. The key idea is to develop a model for the composite interference distribution

outside a fixed cell, as a function of the user position from the cell center. From a numerical perspective, our approach simplifies the simulation study of cellular systems by replacing the sum interference term with an equivalent interference random variable. Then functions like average rate and success probability can be computed through numerical integration rather than Monte Carlo simulation. Unfortunately, the numerical integrals may still require a fair amount of computational power to compute. Consequently, we proposed to approximate the interference distribution by moment matching with the Gamma distribution. We showed how the Gamma distribution could be used to obtain relatively simple expressions for the success probability and ergodic rate, which simplify further when the sum interference power is approximated directly as a Gamma random variable. Simulations showed that with an appropriately chosen size of the guard radius, a reasonable fit between the case of PPP distributed co-channel interference and hexagonal interference was achieved. Further work is needed to better characterize the approximations, e.g. by developing expressions for bounding the error terms.

REFERENCES

- [1] V. Chandrasekhar, J. Andrews, and A. Gatherer, "Femtocell networks: a survey," *IEEE Commun. Mag.*, vol. 46, no. 9, pp. 59–67, Sep. 2008.
- [2] R. Pabst, B. H. Walke, D. C. Schultz, P. Herhold, H. Yanikomeroglu, S. Mukherjee, H. Viswanathan, M. Lott, W. Zirwas, M. Dohler, H. Aghvami, D. D. Falconer, and G. P. Fettweis, "Relay-based deployment concepts for wireless and mobile broadband radio," *IEEE Commun. Mag.*, vol. 42, no. 9, pp. 80–89, Sep. 2004.
- [3] H. Zhu, "Performance comparison between distributed antenna and microcellular systems," *IEEE Journal on Sel. Areas in Commun.*, vol. 29, no. 6, pp. 1151–1163, Jun. 2011.
- [4] V. Chandrasekhar and J. Andrews, "Uplink capacity and interference avoidance for two-tier femtocell networks," *IEEE Transactions on Wireless Commun.*, vol. 8, no. 7, pp. 3498–3509, Jul. 2009.
- [5] J. G. Andrews, F. Baccelli, and R. K. Ganti, "A tractable approach to coverage and rate in cellular networks," *IEEE Transactions on Communications*, vol. 59, no. 11, pp. 3122–3134, Nov. 2011.
- [6] D. Stoyan, W. Kendall, and J. Mecke, *Stochastic Geometry and Its Applications*, 2nd ed. John Wiley and Sons, 1996.
- [7] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, "Stochastic geometry and random graphs for the analysis and design of wireless networks," *IEEE Journal on Sel. Areas in Commun.*, vol. 27, no. 7, pp. 1029–1046, Sep. 2009.
- [8] F. Baccelli and B. Błaszczyszyn, *Stochastic Geometry and Wireless Networks, Volume I - Theory*. NOW Publishers, 2009.
- [9] —, *Stochastic Geometry and Wireless Networks, Volume II - Applications*. NOW Publishers, 2009.
- [10] F. Baccelli, M. Klein, M. Lebourges, and S. Zuyev, "Stochastic geometry and architecture of communication networks," *Telecommunication Systems*, vol. 7, no. 1-3, pp. 209–227, 1997.
- [11] F. Baccelli and S. Zuyev, "Stochastic geometry models of mobile communication networks," *Frontiers in queueing*, CRC Press, vol. 2, pp. 227–244, 1997.
- [12] T. X. Brown, "Cellular performance bounds via shotgun cellular systems," *IEEE Journal on Sel. Areas in Commun.*, vol. 18, no. 11, pp. 2443–2455, Nov. 2000.
- [13] L. Decreusefond, P. Martins, and T.-T. Vu, "An analytical model for evaluating outage and handover probability of cellular wireless networks," Sep. 2010, <http://arxiv.org/pdf/1009.0193v1>.
- [14] V. Chandrasekhar, M. Kountouris, and J. Andrews, "Coverage in multi-antenna two-tier networks," *IEEE Transactions on Wireless Commun.*, vol. 8, no. 10, pp. 5314–5327, Oct. 2009.
- [15] Y. Kim, S. Lee, and D. Hong, "Performance analysis of two-tier femtocell networks with outage constraints," *IEEE Transactions on Wireless Commun.*, vol. 9, no. 9, pp. 2695–2700, Sep. 2010.
- [16] M. Win, P. Pinto, and L. Shepp, "A mathematical theory of network interference and its applications," *Proceedings of the IEEE*, vol. 97, no. 2, pp. 205–230, Feb. 2009.
- [17] J. Ilow and D. Hatzinakos, "Analytic alpha-stable noise modeling in a Poisson field of interferers or scatterers," *IEEE Trans. on Signal Processing*, vol. 46, no. 6, pp. 1601–1611, June 1998.
- [18] X. Yang and A. Petropulu, "Co-channel interference modelling and analysis in a Poisson field of interferers in wireless communications," *IEEE Trans. on Signal Processing*, vol. 51, no. 1, pp. 64–76, Jan. 2003.
- [19] K. Gulati, B. L. Evans, J. G. Andrews, and K. R. Tinsley, "Statistics of co-channel interference in a field of Poisson and Poisson-Poisson clustered interferers," *IEEE Trans. on Signal Processing*, vol. 58, no. 12, pp. 6207–6222, Dec. 2010.
- [20] P. Madhusudhanan, J. Restrepo, Y. Liu, and T. Brown, in *IEEE Global Telecommunications Conference (GLOBECOM '09)*, Dec.
- [21] P. Madhusudhanan, J. Restrepo, Y. Liu, T. Brown, and K. Baker, in *IEEE Global Telecommunications Conference (GLOBECOM '11)*, Dec.
- [22] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Modeling and analysis of K-tier downlink heterogeneous cellular networks," *IEEE Journal on Sel. Areas in Commun.*, vol. 30, no. 3, pp. 550–560, Apr. 2012.
- [23] M. Haenggi and R. K. Ganti, *Interference in Large Wireless Networks*. NOW Publishers, 2010.
- [24] S. Mukherjee, "Distribution of downlink SINR in heterogeneous cellular network," *IEEE Journal on Sel. Areas in Commun.*, vol. 30, no. 3, pp. 575–585, Apr. 2012.
- [25] H.-S. Jo, Y. J. Sang, P. Xia, and J. G. Andrews, "Heterogeneous cellular networks with flexible cell association: A comprehensive downlink SINR analysis," *IEEE Trans. on Wireless Communications*, Jul. 2011, <http://arxiv.org/abs/1107.3602>.
- [26] S. Mukherjee, "Downlink SINR distribution in a heterogeneous cellular wireless network with max-SINR connectivity," in *Proc. of the Allerton Conf. on Comm. Control and Comp.*, Sep. 2011, pp. 1649–1656.
- [27] S. Al-Ahmadi and H. Yanikomeroglu, "On the approximation of the generalized-K distribution by a gamma distribution for modeling composite fading channels," *IEEE Trans. on Wireless Commun.*, vol. 9, no. 2, pp. 706–713, Feb. 2010.
- [28] R. W. Heath Jr. and M. Kountouris, "Modeling heterogeneous network interference," in *Proc. of the Information Theory and Applications Workshop (ITA)*, Feb. 2012.
- [29] A. Hasan and J. G. Andrews, "The guard zone in wireless ad hoc networks," *IEEE Trans. on Wireless Communications*, vol. 6, no. 3, pp. 897–906, Mar. 2007.
- [30] M. Nakagami, "The m distribution: A general formula of intensity distribution of rapid fading," in *Statistical methods in radio wave propagation*, W. G. Hoffman, Ed. Oxford University Press, 1960.
- [31] B. Schmeiser, "Methods for modelling and generating probabilistic components in digital computer simulation when the standard distributions are not adequate: A survey," *ACM SIGSIM Simulation Digest*, vol. 10, no. 1-2, pp. 50–57, Fall-Winter 1977-1978.
- [32] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*. London: Oliver Boyd, 1965.
- [33] P. G. Moschopoulos, "The distribution of the sum of independent gamma random variables," *Ann. Inst. Statist. Math. (Part A)*, vol. 37, pp. 541–544, 1985.
- [34] S. B. Provost, "On sums of independent gamma random variables," *Statistics*, no. 20, pp. 583–591, 1989.
- [35] S. Nadarajah, "A review of results on sums of random variables," *ACTA Applicandae Mathematicae*, vol. 103, no. 2, pp. 131–140, 2008.
- [36] M.-S. Alouini, A. Abdi, and M. Kaveh, "Sum of gamma variates and performance of wireless communication systems over nakagami-fading channels," *IEEE Trans. on Veh. Tech.*, vol. 50, no. 6, pp. 1471–1480, Nov. 2001.
- [37] R. Heath, T. Wu, Y. Kwon, and A. Soong, "Multiuser MIMO in distributed antenna systems with out-of-cell interference," *IEEE Trans. on Signal Processing*, vol. 59, no. 10, pp. 4885–4899, Oct. 2011.
- [38] 3GPP, "Further advancements for E-UTRA physical layer aspects (release 9)," 2010, 3GPP TR 36.814 V9.0.0 (2010-03).
- [39] —, "Coordinated multi-point operation for LTE physical layer aspects (release 11)," 2011, 3GPP TR 36.819 V11.0.0 (2011-09).
- [40] S. B. Lowen and M. C. Teich, "Power-law shot noise," *IEEE Transactions on Information Theory*, vol. 36, no. 6, pp. 1302–1318, Nov. 1990.
- [41] A. Hunter, J. G. Andrews, and S. Weber, "Transmission capacity of ad hoc networks with spatial diversity," *IEEE Trans. on Wireless Communications*, vol. 7, no. 12, pp. 5058–5071, Dec. 2008.
- [42] J. B. Seaborn, *Hypergeometric functions and their applications*. Springer-Verlag, 1991.
- [43] G. E. Andrews, R. Askey, and R. Ranjan, *Special functions*. Cambridge: Cambridge University Press, 2001.
- [44] F. W. J. Olver, *Asymptotics and Special Functions*. AKP Classics, 1965.